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SIMPLE REALIZATIONS OF GENERALIZED MEASUREMENTS IN QUANTUM MECHANICS

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This paper discusses the approach to the analysis of measurements in quantum mechanics which is based on a set of "detection operators" forming a resolution of identity. The expectation value of each of these operators furnishes the counting rate at a detector for any object state that is prepared. "Predictable measurements" are those for which there is a representation in which only one element of each diagonal matrix representing each operator is not zero. A set of commuting detection operators defines the class of "spectral measurements", which may be either predictable or not. An even more general definition of measurement may be given by abandoning the requirement of commutativity of the detection operators. In this case one cannot define an observable which corresponds to a single self-adjoint operator, which violates the standard theory of quantum mechanical measurement. Simple experimental realizations of each of these classes of measurement are suggested.

Key words: quantum measurements, observables, distinguishability of states, insolubility proofs, polarization experiments.

1. PREDICTABLE MEASUREMENTS

Several measurements performed on identically prepared quantum systems in general do not yield one single result, but a spread of outcomes. A measurement has been called *predictable* [1] if each of the possible outcomes of the measurement can be obtained with certainty from *some* preparation of the initial object state. Each object state that yields a predictable outcome is represented by a unit vector $|\phi_i\rangle$ in the

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object Hilbert space $\mathcal{H}_o$, which we assume for simplicity to be finite-dimensional. The set $\{ | \phi_i \rangle \}$ forms an orthonormal basis spanning $\mathcal{H}_o$. Predictable measurements include both repeatable and non-repeatable measurements, which have also been respectively called measurements of the "1st" and "2nd kind" [2].

According to the standard approach to quantum measurements [3,4], the measurement of any physical quantity can be associated with a self-adjoint operator $\hat{O}$ acting on $\mathcal{H}_o$, and the possible outcomes of an individual measurement are the eigenvalues γ_i of $\hat{O}$, each of which corresponds to an eigenstate $| \phi_i \rangle$, satisfying:

$$\hat{O} \, | \phi_i \rangle = \gamma_i | \phi_i \rangle . \qquad (1)$$

The measured physical quantity which is associated to a self-adjoint operator is called an "observable".

Defining $\hat{P}[\phi_i] \equiv | \phi_i \rangle \langle \phi_i |$ as the projection operator onto the subspace spanned by $| \phi_i \rangle$, the operator $\hat{O}$ corresponding to the observable being measured can be written, according to the spectral theorem [4], as $\hat{O} = \sum_i \gamma_i \hat{P}[\phi_i]$. For a generic initial object state $\sum_i a_i | \phi_i \rangle$, the probability for each outcome γ_i is given by $| a_i |^2$.

As a trivial example of a predictable measurement, consider the measurement of linear polarization of a light beam shown in Fig. 1. The beam passes through a birefringent analyzer A such as a Wollaston prism and is separated into two channels, each of which falls on a detector, either D_1 or D_2. It is easy to prepare a beam such that only one of the detectors triggers. One can prepare such a beam by passing it through a dichroic polarizer F oriented at an appropriate angle, which we will denote by 0°, and which depends on the orientation of the analyzer. A beam that will only trigger the second detector can be prepared by initially passing it through a filter oriented at 90°. Such pure beams may be described by orthogonal states $| \phi_1 \rangle$ and $| \phi_2 \rangle$, which span the two-dimensional object Hilbert space $\mathcal{H}_o$. The observable being measured in this experimental setup is represented by $\hat{O} = \gamma_1 \hat{P}[\phi_1] + \gamma_2 \hat{P}[\phi_2]$, where the eigenvalues γ_i indicate linear polarizations at 0° and at 90°.

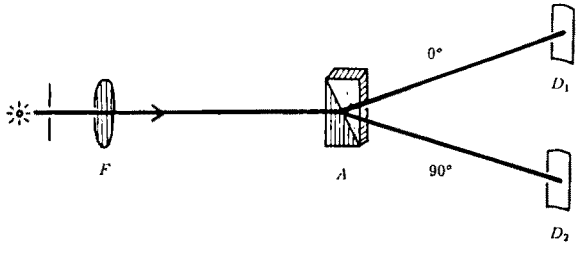

Fig. 1. Example of a predictable measurement in optics.

2. THE DETECTION OPERATOR

How is the association between a physical quantity (an observable) and an operator established? A measurement involves basically three stages. The object system is initially prepared (by some procedure) in a certain state, which for generality can be represented by a density operator $\hat{W}$. The system then interacts with parts of a measuring apparatus, such as a target or a magnetic field. After interaction, detection or registration takes place. Registration usually involves several spatially distributed detectors, which we will label by the letter u (for a detecting screen, each of its points can be considered a separate detector).

If the object system corresponds to a single particle, we then have an "individual measurement", and at most one of the detectors will trigger. Usually, however, the preparation procedure generates many particles, which under favorable conditions can be collectively described by a density operator, and we have an "ensemble of measurements". In this case each detector registers a *counting rate*, which after normalization furnishes a probability distribution $\mu_W(u)$ for detection at each detector u, for a system prepared in the state W.

A "statistical model" of quantum mechanics [5] views a measurement as an affine mapping from the set of states $\hat{W}$ onto the set of probability distributions $\mu_W(u)$. To express such a mapping, one can define in $\mathfrak{H}$ an operator $\hat{M}(u)$ for each detector, so that the expectation value of $\hat{M}(u)$ over the initial object state $\hat{W}$ furnishes the probability $\mu_W(u)$ for a count at detector u:

$$\mu_W(u) = \mathrm{Tr}\,(\hat{W}\,\hat{M}(u)). \tag{2}$$

We will call $\hat{M}(u)$ "detection operators". Each $\hat{M}(u)$ is positive, *i.e.*, $\langle \phi \mid \hat{M}(u) \mid \phi \rangle \geq 0$ for any state vector $\mid \phi \rangle$ (number of counts or probabilities are non-negative numbers), and self-adjoint. Each is normalized so that its trace is 1. If all the prepared particles are detected, one can show [5] that the set of detection operators forms a *resolution of identity*:

$$\sum_u \hat{M}(u) = \hat{I}. \tag{3}$$

One can express the detection operators in terms of the evolution operators $\hat{U}$, which were introduced by von Neumann to describe the interaction between object and apparatus during measurement [6]. The resolution of identity (3) is equivalent to the requirement that these evolution operators be *unitary* [7].

A predictable measurement (which has also been called a *simple measurement* [5], or, in the case in which there are only two detection operators, a *simple test* [8] or a *decision effect* [9]) is one in which each detection operator $\hat{M}(u)$ corresponds to a projection operator $\hat{P}[\phi_i]$ onto

the subspace spanned by an eigenstate $|\phi_i\rangle$, such as in (1). This implies not only that $\hat{M}(u)^2 = \hat{M}(u)$, but also [5] that $\hat{M}(u)$ and $\hat{M}(u')$ are orthogonal projections:

$$\text{For any } u \neq u', \quad \hat{M}(u)\hat{M}(u') = 0. \tag{4}$$

In the example of predictable measurement of the previous section, the detection operators are represented in a simple manner by the following matrices in the basis of representation $\{|\phi_1\rangle, |\phi_2\rangle\}$:

$$\hat{M}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{M}(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5}$$

For any predictable measurement, there is always a representation in which the detection operators can be expressed by diagonal matrices with only one non-zero element.

3. NON-PREDICTABLE SPECTRAL MEASUREMENTS

The standard approach to measurements in quantum mechanics hinges on conditions (3) and (4), which define *orthogonal* resolutions of identity (or projection-valued measures). It can be argued, however, that actual physical measurements are never predictable, due to non-ideal efficiencies of analyzers [8]. Furthermore, there is no a priori reason why there could not be *non-orthogonal* resolutions of identity (positive operator-valued measures, also called "semispectral resolutions" [10]), corresponding to non-predictable *generalized measurements*.

In 1940 M.A. Naimark considered this possibility, but he proved a theorem [11-14] which in a sense reduces non-predictable measurements to simple ones. Given a non-orthogonal resolution of identity for $\hat{M}(u)$ in $\mathcal{H}_0$, satisfying (3) but not (4), one can introduce an auxiliary Hilbert space $\mathcal{H}_\zeta$, such that there is always an orthogonal set of projection operators in $\mathcal{H}_0 \otimes \mathcal{H}_\zeta$ satisfying (3) and (4), and which reduce to $\{\hat{M}(u)\}$ when a partial trace is performed over states in $\mathcal{H}_\zeta$ (*i.e.*, when the operators are projected onto $\mathcal{H}_0$). The standard approach can in this way be said to apply even when the statistical correlations between state preparations and counting rates at the different detectors give rise to operators $\hat{M}(u)$ which are not orthogonal projectors. One can argue that what is really being measured in this case does not correspond to non-orthogonal operators in $\mathcal{H}_0$, but to a set of orthogonal ones in $\mathcal{H}_0 \otimes \mathcal{H}_\zeta$.

In spite of the mathematical correctness of the theorem, the above strategy for saving the standard approach appears to be abstract and ad

hoc: in concrete generalized measurements there is probably no physical means of establishing the existence of an auxiliary system which behaves in the required way.

The first to introduce general resolutions of identity in quantum measurement theory were Davies and Lewis [15] in 1970. They considered the description of a succession of measurements on an object system, and were concerned with the fact that if two observables corresponding to non-commuting operators were measured one after the other, then the probabilities of detection of the overall process could not be obtained from a measurement of a single observable in the standard theory [16]. They therefore extended the definition of what counts as a measurement to positive operator-valued measures, and also generalized the notion of "observable" to include the "conditioned observables" associated with non-orthogonal resolutions of identity.

Let us consider an example which illustrates a succession of simple measurements of observables associated with operators which do not commute, and see that the detection operators are not orthogonal. We will consider a beam of light which passes by a sequence of polarization analyzers in non-orthogonal orientations, with all the separated beams falling on one of two detectors.

A beam of light is separated at the birefringent prism A_1, oriented at $\theta=0°$ (see Fig. 2). The "top component" $|\phi_1\rangle$ which is polarized at 0° falls on detector D_1. The "bottom component" $|\phi_2\rangle$ polarized at 90° falls on analyzer A_2 oriented at $\theta=45°$, and so is split into a component polarized at 45°, which also is collected at detector D_1, and a component polarized at 135°, which falls on detector D_2. The setup is such that the two beams falling on D_1 do not interfere. Strictly speaking, the situation depicted does not correspond to a succession of measurements (as considered by Davies and Lewis), but to a single measurement (an ensemble of measurements), since detection of a photon only occurs after it has passed the set of analyzers.

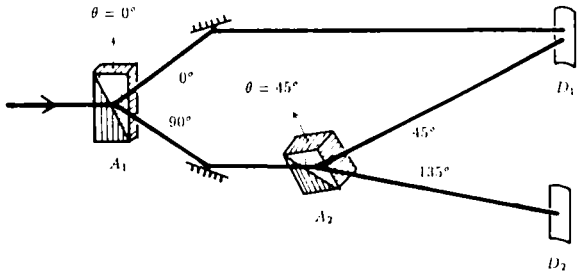

Fig. 2. Measurement involving a succession of analyzers oriented at non-orthogonal directions.

A brief calculation furnishes the following detection operators, in the representation $\{ |\phi_1\rangle, |\phi_2\rangle \}$:

$$\hat{M}(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \hat{M}(2) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \qquad (6)$$

These operators are clearly not orthogonal, violating (4). We therefore have a measurement the outcomes of which cannot be described by a simple measurement in the standard approach. The measurement is not predictable, since there is no state preparation that will trigger with certainty detector D_2, although it is simple to prepare a state that will trigger D_1 for sure.

We could complicate the setup a little further by introducing a third analyzer A_3 oriented at $\theta = 30°$ in the path of the "top component" (Fig. 3), and detecting the resulting beams polarized at 30° and at 120°, respectively, at D_1 and D_2. The resulting detection operators are represented by:

$$\hat{M}(1) = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \hat{M}(2) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \qquad (7)$$

Once again, the matrices representing the detection operators are not orthogonal, but they commute. Although the measurement violates (4), could one construct an operator $\hat{O}$ which satisfies (1), so that a single observable (in the sense of the standard approach) could be

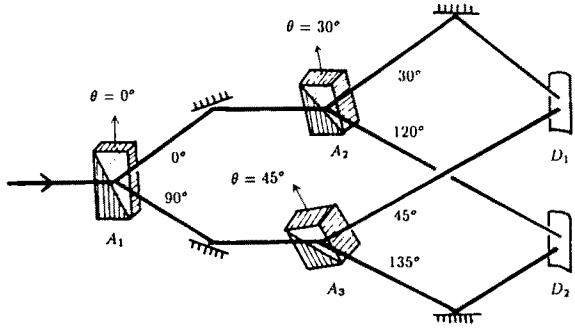

Fig. 3. Realization of a non-predictable spectral measurement.

associated to the measurement? The answer is yes. For the measurement given by (7), the operator which satisfies (1) can be expressed in terms of the detection operators $\hat{M}(u)$ as follows:

$$\hat{O} = (2\gamma_1 - \gamma_2)\hat{M}(1) + (-2\gamma_1 + 3\gamma_2)\hat{M}(2) = \gamma_1\hat{O}_1 + \gamma_2\hat{O}_2. \quad (8)$$

Non-predictable measurements with commuting detection operators (which for P=2 have been called *mixed tests* [8] or *effects* [9]) can therefore be said to be described by the standard approach, although they violate (4). If the number of detectors P is not greater than the dimensionality K of the object Hilbert space $\mathcal{H}_o$ (in the above case, P=K=2), the measurement may be termed *non-redundant*. A non-redundant measurement will be called a *spectral measurement* if the detection operators form a mutually commuting set:

$$\text{For any } u,u', \quad \hat{M}(u)\hat{M}(u') - \hat{M}(u')\hat{M}(u) = 0. \quad (9)$$

By means of a spectral measurement one is able to "distinguish" two object states $|\phi\rangle \equiv \Sigma\, a_i |\phi_i\rangle$ and $|\phi'\rangle \equiv \Sigma\, a'_i |\phi_i\rangle$, which in the representation $\{|\phi_i\rangle\}$ (of the eigenstates of the operator $\hat{O}$ corresponding to the observable being measured) are such that $|a_i|^2 \neq |a'_i|^2$ for at least one of the indices i. A precise definition of such "$\hat{O}$-distinguishability" for spectral measurements was provided by Fine [6], in his definition of "W_*-measurements" (which are equivalent to spectral measurements). The dimensionality K' of the largest subspace of $\mathcal{H}_o$ whose vectors can be distinguished in this way defines the property of *K'-distinguishability*. If K is the dimension of $\mathcal{H}_o$, then a measurement such as the above which is K-distinguishing may be termed *non-degenerate*.

4. NON-SPECTRAL MEASUREMENTS

A *non-spectral measurement* would involve a set of detection operators that are not mutually commuting. It is possible to arrange such a measurement simply by splitting the incoming beam into two spatially separated parts, which in the optical case can be realized with a half-silvered mirror. The observables measured in each part would be chosen so that the operators corresponding to them would not commute. This in fact constitutes the most efficient way of determining the state of an object system [17].

It turns out that for a 2-dimensional object space, there is no non-spectral measurement which is non-redundant (K=P=2) and for which all of the prepared beam is detected, satisfying the resolution of identity (3). In higher-dimensional cases, however, there exist in principle such unitary spectral measurements in which no part of prepared beam is lost.

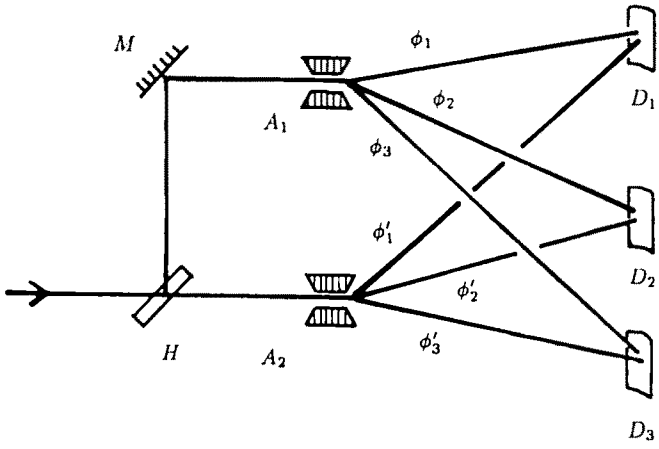

Fig. 4. Realization of a non-spectral measurement with spin-1 particles.

As an example of a 3-dimensional object space, consider a beam of spin-1 particles, as illustrated in Fig. 4. We assume that the beam can be spatially separated at H into two parts irrespective of its spin state, and that each part passes through a Stern-Gerlach analyzer which coherently separates the beam into three spin components. The two analyzers used are oriented in different angles, so that the "top" analyzer A_1 separates the beam into the spin eigenstates $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, while the "bottom" one A_2 separates the beam into orthogonal states $|\phi'_1\rangle$, $|\phi'_2\rangle$, $|\phi'_3\rangle$, given for instance by:

$$
\begin{aligned}
|\phi'_1\rangle &= \tfrac{1}{\sqrt{2}}|\phi_1\rangle + \tfrac{1}{\sqrt{2}}|\phi_3\rangle, \\
|\phi'_2\rangle &= \tfrac{1}{2}|\phi_1\rangle + \tfrac{1}{\sqrt{2}}|\phi_2\rangle - \tfrac{1}{2}|\phi_3\rangle, \\
|\phi'_3\rangle &= -\tfrac{1}{2}|\phi_1\rangle + \tfrac{1}{\sqrt{2}}|\phi_2\rangle + \tfrac{1}{2}|\phi_3\rangle.
\end{aligned}
\tag{10}
$$

The different separated beams fall on one of three detectors, as shown in Fig. 4. The computation of each detection operator yields:

$$
\hat{M}(1) = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}, \quad \hat{M}(2) = \begin{bmatrix} \frac{1}{8} & \frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ \frac{1}{4\sqrt{2}} & \frac{3}{4} & -\frac{1}{4\sqrt{2}} \\ -\frac{1}{8} & -\frac{1}{4\sqrt{2}} & \frac{1}{8} \end{bmatrix},
$$

$$\hat{M}(3) = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ -\frac{1}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4\sqrt{2}} \\ -\frac{1}{8} & \frac{1}{4\sqrt{2}} & \frac{5}{8} \end{bmatrix} .$$

The above set of detection operators is non-redundant and K-distinguishing (for K=3) but not pairwise commuting, so the measurement is clearly non-spectral. Pairs of object states can be distinguished, but not according to Fine's criterion. Contrary to the case of spectral measurements, one can in general distinguish states $|\phi\rangle \equiv \Sigma\, a_i |\phi_i\rangle$ and $|\phi'\rangle \equiv \Sigma\, a'_i |\phi_i\rangle$ for which the members of each pair a_i, a'_i, differ only by a relative phase (*i.e.*, $|a_i|^2 = |a'_i|^2$), whatever the orthonormal basis $\{|\phi_i\rangle\}$ may be. On the other hand, for non-spectral measurements, a convenient choice of basis states may make a pair of indistinguishable states $|\phi\rangle$, $|\phi'\rangle$, differ only by relative phases, but in general there will be other states $|\phi''\rangle$ that are also indistinguishable from the pair, but for which $|a_i|^2 \neq |a''_i|^2$ for some i. These properties violate Fine's criterion of distinguishability.

5. CONCLUSIONS

Non-spectral measurements are adequately described by the positive operator-valued measures introduced by Davies and Lewis and by Holevo, but it is clear that they cannot be described by the standard approach to measurements in QM, which is restricted to commutative resolutions of identity. There is no criterion for finding an operator $\hat{O}$ with a unique set of eigenstates that satisfies (1) and that can be associated with the non-spectral measurement. Thus the notion of "observable" cannot be applied to non-spectral measurements, at least as it is defined by the standard theory of measurement. The notion that is more suitable to the generalized class of measurements is that of a set of detection operators which form a resolution of identity (a positive operator-valued measure), which has been called a "unorthodox measurement" [10].

One consequence of this conclusion bears on the so-called "insolubility proofs of the measurement problem" [18]. The most general insolubility proof was developed by Fine [19] for his W_a-measurements, which are equivalent to spectral ones. Non-spectral measurements are therefore not encompassed by any of the insolubility proofs presented in the literature. *Could there be a non-spectral measurement that is "soluble"?*

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