Towards a Modal Logical Treatment of Quantum Physics

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Abstract

The question of which is the logic that underlies quantum physics does not have an absolute answer, but only in relation to a conventional choice of interpretation (in the sense used in the study of scientific theories). Most of the interpretations that have been offered work within the framework of classical logic. In contrast to these, we examine the corpuscular interpretation which is assumed in the application of non-distributive logic (section 3). The experiment in which single photons pass through a Mach-Zehnder interferometer is examined, indicating the difficulty of employing a realist corpuscular interpretation in this context. One way to save this interpretation would be to use non-distributive logic to analyze the experiment, but this is not satisfactory (section 4). However, the use of an alethic modal logic solves the problem, blocking the argument that put the aforementioned corpuscular interpretation into difficulty. In the discussion of the conceptual problems involved, we suggest that a stochastic corpuscular interpretation is well adapted to this logical description. The project of extending this modal logical approach to other experiments in quantum physics, and of providing a rigorous logical treatment, is left open.

Keywords: quantum mechanics, quantum logic, modal logic, interferometer, interpretation

1 Interferometer for a Single Quantum

A didactical way of presenting quantum theory [11] is by means of the Mach-Zehnder interferometer for a single quantum[1]. The "quantum" in question is usually a photon (associated to light), but one may also perform a version of the experiment with a single electron.

The apparatus is shown in Fig. 1. A laser emits a beam of "monochromatic" light, that is, a beam that may be represented by a simple sine wave. The beam passes through a half-silvered mirror S_1, which divides the beam into a transmitted component (A) and a reflected one (B). Each component is then reflected in mirrors E_1 and E_2, and cross again at the half-silvered mirror S_2, from where they head towards detectors D_1 and D_2. What happens?

Since each component splits in half in S_2, one might expect that each detector would measure 50% of the initial beam. But that is not what happens! If the distance traveled by each component is exactly the same, and if the mirrors are perfectly aligned, one observes that 100% of the original beam falls on D_1 and 0% on D_2 (if this does not occur, then one just has to slowly turn the thin glass H until it occurs)! Fig. 1 shows why this happens: the two components that head towards D_2 have relative phase shifts that cancel each other. This phenomenon is known as *destructive superposition.*

[1] One usually refers to a single "particle", but this term is associated with the idea of an entity that describes a trajectory. However, the existence of such an entity is exactly the point which is under discussion. Thus, it is preferable to use a term such as "quantum", which might be understood as a "detected particle", with no commitment to what happens before the detection. The term "particle" will only be used when a corpuscular interpretation is presupposed.

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The description up to here is in accordance with classical physics. To render the experiment quantum mechanical, one must prepare beams of light of very low intensity and construct detectors of high sensibility (for simplicity, considered perfectly efficient). Under these conditions, something unexpected to the eyes of classical physics happens: light is detected in discrete and indivisible units of energy, known as *photons*. In accordance with the classical situation, all the photons are detected in D_1 and none in D_2. In the 1980's, such an experiment was performed with exactly one photon entering at a known time.

The question to be now posed is the following. In the interferometer of Fig. 1, *after the photon passes through* S_1, *but before it falls on* S_2, *in which path is it located, in* A *or in* B?

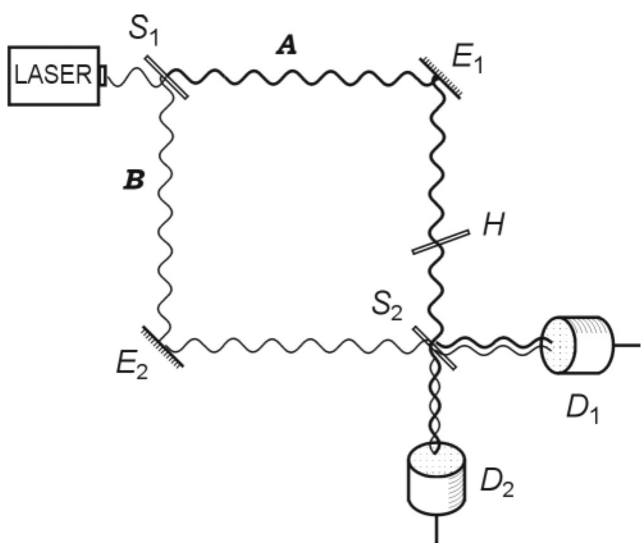

Figure 1: The Mach-Zehnder interferometer.

Suppose that the photon is located in A, and not in B. This may be experimentally implemented by removing the half-silvered mirror S_1 from its place. In this case, the photon falls on S_2, and may be detected *either* in D_1 (50% probability) *or* in D_2 (50%), without the interference of components. If we suppose that the photon is initially located in B, and not in A (implemented by replacing S_1 by a totally reflecting mirror), there would also be a 50% chance of detecting it in D_2.

Now, if the photon were *either* in A *or* in B, we would still have a 50% probability of detecting the photon in D_2. This follows from the definition of the logical connective "or": if in A it is 50% and in B it is 50% (two situations that are exclusive), then in "A or B" it must be 50%. However, we have seen in the above experiment, for a single photon, that the probability of the photon arriving in D_2 is not 50%, but 0%! Therefore, *the statement that the photon is* either *in A (there being nothing in B)* or *in B (there being nothing in A) is false*!

This is incredible! If light is detected in the form of particles (quanta), one would expect that such particles would exist during the propagation of light, following well defined (even if unknown) paths. However, we have arrived at the conclusion that the photons do not follow well defined trajectories!

2 Interpretations for Explaining the Paradox

To solve this conceptual problem, one must appeal to one of different "interpretations" of quantum theory. There are at least four basic interpretations which may be used (see [11], ch. II): three of these are *realist*, postulating the continuous existence of real entities such as waves, particles, or both, and the fourth is *positivist* (instrumentalist), restricting its description only to observed events.

To give an example of how an interpretation can solve the aforementioned paradox, consider the view that defends a purely undulatory interpretation. According to this "wave" interpretation, point particles do not exist, but only waves. These, however, can be squeezed into compact "wavepackets", the detection of which may be restricted to a quite small region of space (thus "looking like" a particle). In the experiment in which exactly *one* photon enters the interferometer, what is happening – according to this view – is the entrance of a pulse with sufficient energy to trigger the detector only once. After the first beamsplitter, this interpretation postulates that the photon (monophotonic wavepacket) is spatially divided into two symmetric parts (related by a fixed relative phase, which explains the destructive superposition that happens later on). However, these parts maintain some sort of intimate contact, even when separated by a great distance, so if one of them triggers a detector (in a setup different from Fig. 1, in which detectors are inserted in paths A and B), the other will necessarily *not* trigger the other detector. This strange property, which arises in the wave interpretation, is called the "collapse" of the wavepacket. Such a process would be practically instantaneous, involving a "passion at a distance" ([15], p. 133), which is strange to the intuitions that we have regarding the propagation of signals at finite velocities.

In spite of these and other strange features or "anomalies" (as Reichenbach called them, see [14], p. 32), the wave interpretation is consistent, explaining satisfactorily all quantum mechanical phenomena, and – more importantly for us, here – using *classical logic*. The realist dualist interpretation of David Bohm also uses classical logic, in spite of its own collection of conceptual anomalies.

Given that none of the interpretations of quantum theory is greatly superior to the others, it is clear that this theory *does not require* an abandonment of classical logic in favor of a "quantum logic". Notwithstanding, many authors defend that the logic that underlies quantum mechanics is a non-distributive logic (in the finite-dimensional case which we are considering, distributivity must be replaced by "modularity"). Why do they claim this? Because they are adopting (sometimes without noticing) a specific interpretation which requires a non-distributive logic: in general, a realist corpuscular interpretation (that is, an interpretation which assigns at all times definite values to most or all observables, including position).

3 Non-Distributive Logic

The discovery that the calculus of propositions of quantum mechanics is a non-classical logic goes back to the pioneering work Garrett Birkhoff & John von Neumann in 1936 (see [7], ch. 8), in which they showed that the following property of distributivity is in general *not* valid:

$$C \wedge (A \vee B) \ \vdash \ (C \wedge A) \vee (C \wedge B). \tag{1}$$

The interpretation of the logical connectives "$\wedge$" ("and") and "$\vee$" ("or") arises from the properties of projectors in Hilbert space: the first represents the *intersection* (for example, the intersection between two different axes has value zero) and the second the *direct sum* (the direct sum of two different axes is a plane). This latter interpretation is different from the notion of disjunction we are familiar with in classical particle physics. What justifies interpreting these properties of projectors as logical connectives? Jauch [8] proposed an experimental interpretation, for intersection and direct sum, by means of filters (such as polarizers, for light), but such experimental operations remain distant from the usual meanings associated with the logical connectives "and" and "or".[2]

In the 1960's, David Finkelstein and Hilary Putnam argued that the problem of what is the logic of the world should receive an empirical answer, in a manner analogous to the problem of what is the geometry of space. Putnam presented a simple example, involving the double slit experiment, which furnished a meaning to the logical connectives "$\wedge$" and "$\vee$". As mentioned by Gibbins ([5], ch. 10), Putnam's proposal defended a version of realism in which each corpuscle would have an exact position and momentum. However, Gibbins and others have shown that Putnam's example was faulty.

There is a version of Putnam's argument that involves a spin-$\frac{1}{2}$ quantum ([6], [3], p. 201-2), and which we reproduce as follows.

i) If a spin-$\frac{1}{2}$ quantum is prepared in the eigenstate $|+z\rangle$, the following sentence P is true: "The spin component in the z direction is up".

ii) If we now measure the spin component in the x direction, the measured value is either $+$ (up) or $-$ (down). This may be expressed by the statements Q and R, where Q is "The spin component in the x direction is up" and R "The spin component in the x direction is down". Consequently, the proposition "$Q \vee R$" is true.

iii) Since "P" is true, and "$Q \vee R$" is true, we can assert that "$P \wedge (Q \vee R)$" is true.

iv) By the uncertainty principle, there cannot be simultaneously well-defined values for the spin component in the z direction and in the x direction.

v) Accepting the statement of the uncertainty principle in the previous step, to state "$P \wedge Q$" would be false; similarly, "$P \wedge R$" would be false.

vi) From the previous step, one may assert that "$(P \wedge Q) \vee (P \wedge R)$" is false.

vii) Finally, accepting steps *ii*, *iv* and *v*, one arrives at the conclusion, comparing steps *iii* and *vi*, that "$P \wedge (Q \vee R)$" is not equivalent to "$(P \vee Q) \wedge (P \vee R)$", contrary to the situation in classical logic. In other words, distributivity is violated in quantum logic.

What is not clear in the preceding argument are which interpretative assumptions are being adopted. In step *ii*, everyone agrees that a measurement of the spin component gives either the value $+\frac{1}{2}$ or the value $-\frac{1}{2}$. But to infer from this that the value possessed by the quantum right *before* the measurement is either $+\frac{1}{2}$ or $-\frac{1}{2}$ (as is done in sentences Q and R) involves an additional hypothesis, known as the assumption of "faithful measurements" ([13], p. 89). The wave interpretation, that was briefly presented in section 2, rejects this assumption, since it pictures the quantum entity as an extended wave, which generally does not possess well-defined values for position and spin component before a measurement. According to the wave interpretation, sentence Q is false and R is false, so that "$Q \vee R$" is false, contrary to *ii*.

[2] It is true that disjunction may be "interpreted" in similar way in classical and quantum logic, respectively as the least upper bound of sets (union) or subspaces (join). But the issue being raised is how to "physically interpret" a proposition involving disjuncts.

On the other hand, the class of interpretations that may be called corpuscular tend to accept the assumption of faithful measurements. This is the case, for instance, of the statistical ensemble interpretation of Ballentine [1] (see discussion in [12]). Curiously, this approach also claims that a single particle possesses simultaneously well-defined values for the spin components along the x and z directions! Therefore, for this interpretation, the sentence "$P \wedge Q$" would be true, contrary to step *v*. The uncertainty principle would be a statistical limitation, reflecting the impossibility of preparing identical microscopic states in a controlled manner, and not an ontological limitation on possessed values. In this way, a proponent of the ensemble interpretation would reject *v*, while the wave interpretation would accept it.

The upshot of the discussion is that the use of (non-distributive) quantum logic for describing the microscopic world (or the measurements thereof) is consistent and correct, but it is different from other interpretations. In the example just seen, quantum logic accepts the assumption of faithful measurements (in *ii*) (which is typical of corpuscular interpretations) and the ontological version of the uncertainty principle (in *v*), but rejects the distributive law (in *iv*).

This conclusion is quite different from the assertion that the foundations of quantum mechanics *requires* a change in logic, or that the underlying logic of the microscopic world is non-distributive. On the contrary, the question of which is the underlying logic depends on the physical interpretation which is assumed (this discussion is based on [10], pp. 233-6).

4 Formalization of the Interferometry Argument

Let us now return to the discussion of the Mach-Zehnder interferometer for a single photon. Could a corpuscular interpretation explain this experiment? The situation is very difficult, since without postulating the existence of waves, one cannot make use of the concept of "destructive interference". Landé [9] developed an explanation for diffraction of electrons by a crystal, which could be adapted to our problem. When the photon passes through the position of S_1, in Fig. 1, the information about what kind of mirror is in position (a totally reflecting mirror, a half-reflecting one or a piece of transparent glass) would have to be sent to the half-silvered mirror S_2, in such a way that the latter could behave appropriately, either transmitting or reflecting the photon. Such a mechanism, however, is completely foreign to the known laws of physics (although it does not involve velocities that are greater than the speed of light), so that it should be discarded. Is there any hope for an exclusively corpuscular approach? Perhaps yes, if the "anomaly" passes from the physical phenomenon to the logic which rules the quantum world.

Let us express the interferometer argument (section 1), that brings difficulties to the corpuscular interpretation, in a formal manner. Sentence A states that "the photon is in A", while B says that "the photon is in B". Consider also sentence D_2, which states that "there is a non-zero probability of detection in D_2". One could then set up the following argument:

$$\begin{array}{ll} \textbf{P1} & A \vdash D_2 \\ \textbf{P2} & B \vdash D_2 \\ \textbf{P3} & \vdash A \vee B \\ \hline & \vdash D_2 \end{array} \qquad (2)$$

The argument follows directly from the rule of "∨-elimination" (elimination of disjunction), valid in classical logic. However, we have seen that the conclusion of this (valid) argument is false, since the probability of detection in D_2 is zero (Fig. 1). This means that one of the premisses must be false, and the natural candidate is P3: it is therefore not true that, inside the interferometer, the photon (understood as a corpuscle without any associated wave) is in A *or* is in B. The corpuscular interpretation fails, assuming classical logic.

Let us now try to avoid the valuation of P3 as false, making use of non-distributive quantum logic, examined in section 3. Does the violation of the property of distributivity (1) block argument (2)? The answer is no. Gibbins ([5], p. 135) developed a system of natural deduction for quantum logic, in which the rule for "∨-elimination" holds for the restricted case of (2) (there is a more general case of the rule that does not hold). In other words, even if we have recourse to non-distributive logic, the argument turns out to be valid and the conclusion false, so that premiss P3 still has to be denied: it is false that, inside the interferometer, the photon is in A *or* is in B.

This attempt to use non-distributive logic to block the argument failed. But, in fact, such an attempt did not use quantum logic in the usual sense, since sentence D_2 of premisses P1 and P2 involves a *probability*, while each well-formed sentence of non-distributive quantum logic should be associated to a subspace of the Hilbert space of the system being examined. In other words, (2) is not an argument of quantum logic[3]. How should we proceed to offer a non-classical solution to the corpuscular interpretation, in this case?

5 An Interpretation based on Modal Logic

In the logical description that was made in the previous section, two different kinds of propositions were used. "A" says that the photon is in A, but "D_2" says there is a *probability* (different from 0 and 1) of detecting a photon. It would be important to work with propositions of the same kind. We will do this by abandoning an ontological language and applying modal operators to sentences that express the outcomes of measurements.[4]

So let us introduce the (intentional) modal operators[5] of necessity $\Box$ and possibility $\Diamond$. Formula "$\Box A$" will be interpreted as "a measurement at position A would *necessarily* detect the presence of a photon". If in fact there is a photon at that location and a perfectly efficient detector, then it will be true that "$\Box A$". On the other hand, if the pulse containing a single photon passes through the half-silvered mirror S_1, there will be a probability $\frac{1}{2}$ of it being detected at A. Instead of attributing the truth value "indeterminate" to the sentence "a photon is at A" (as would be done by Reichenbach [14], p. 144, in his proposal for athree-valued quantum logic), we will say "$\Diamond A$", that is, "a measurement at position A

[3] One could attempt to overcome this limitation by considering that the square integrable functions on any probability space form a Hilbert space.

[4] It is also interesting to use temporal modal logic to express the contention of Wheeler ([17], p. 194) that "the past has no existence except as it is recorded in the present." The interpretation used by Wheeler is a somewhat realist version of the orthodox interpretation (see [11], sections III.3 and 4). In this case, one may argue that his interpretation violates the prohibition of backward causation, expressed by the postulate of temporal logic which states that "the truth of X implies that it will be true, for all future times t, that at some time in the past (of t) X was true".

[5] Interpretations of quantum theory that are based on modal logic have already been suggested in the literature, but are different from the one proposed here. Van Fraassen ([16], p. 336) offered a modal interpretation for mixtures, introducing two forms of attributing a state to a system: "X is *necessarily* in state ϕ" and "X is *actually* in state ϕ". The so-called "modal interpretations of quantum mechanics", that Kochen, Dieks, Healey and others have developed since 1985, has similarities with van Fraassen's approach, from which they inherited the name — since they also consider *possible* attributions of values for each of the interacting subsystems of an entangled system (see [4]). However, they do not make explicit use of the operators of an alethic modal logic.

would *possibly* detect the presence of a photon".

Considering measurements at positions A (at time t_0), B (at the same time t_0) and D_2 (at some later time t_1) (of Fig. 1), we propose the following argument:

$$\begin{array}{ll} \textbf{P1} & \Box A \vdash \Diamond D_2 \\ \textbf{P2} & \Box B \vdash \Diamond D_2 \\ \textbf{P3} & \vdash \Box(A \vee B) \\ \hline & \vdash \Diamond D_2 \end{array} \tag{3}$$

Premiss P1 states that "if the situation is such that a measurement at A necessarily would detect a photon, then (without carrying out this measurement) a subsequent measurement at D_2 would possibly detect the photon". Premiss P2 is analogous. P3 asserts that "a measurement involving a detector at A and another one at B would necessarily register a photon *either* in the detector at A *or* in the detector at B". All of these premisses seem true, according to our intuition. What about the conclusion? We've already seen that it is not possible to detect a photon at D_2, so the conclusion is false.

Is the argument valid? Contrary to argument (2), this argument involving alethic modal logic is not valid. The following inference is not valid in modal logic:

$$\Box(A \vee B) \ \vdash \ \Box A \vee \Box B. \tag{4}$$

Necessity does not distribute over disjunction. Hence, one cannot reduce the premisses of (3) to the premisses of an argument by "$\vee$-elimination". There is a similarity between (4) and the violation of distributivity in (1): the role of the conjunction with C is now played by the modal operator. However, modal logic invalidates argument (3), while non-distributive logic does not block (2).

Therefore, to conclude, one may say that a strictly corpuscular interpretation may be upheld if it is coupled to a modal logic as described above.[6]

6 Conceptual Discussion

Although we have formally "saved" a strictly corpuscular interpretation of quantum theory by imposing modal logic, one should still discuss whether such a solution is conceptually satisfactory. More specifically, how would a realist corpuscular interpretation explain the violation of (4)?

The antecedent of (4) states that "if a detector were placed at A and another at B, necessarily one (and only one) of them would trigger". The consequent of (4) states that "if a detector were placed at A, it would necessarily trigger, or if a detector were placed in B, it would necessarily trigger". Why wouldn't these assertions be equivalent?

A physical ontology that explains the difference between these assertions is a *stochastic* worldview. If, in fact, the detection of the particle in A or in B were a stochastic (indeterministic) event, without visible or hidden determinants, then the consequent of (4) would clearly be false, since a detector placed in one of the locations would never trigger in a *neces-*

[6] This is, of course, only a suggestion for a direction of research and not the description of a logical system, with explicit axioms, rules, logical interpretations and metalogical results. A name such as "quantum modal logic" could be given to the approach, but such an expression is fraught with ambiguities.

sary manner (since the event is stochastic and there are two equally probable possibilities). On the other hand, the antecedent of (4) would be true. Summarizing, the interpretation sketched here is a stochastic corpuscular view obeying a modal logic.

We should, however, be careful in this conceptual discussion. A crucial point is the interpretation given to the formula "$\Box(A \vee B)$", the antecedent of (4). Why have we interpreted it as saying that "if a detector is placed in A and another in B, one of them would necessarily trigger"? The motivation here is similar to what happens in non-distributive quantum logic: the connective "or" is associated to the direct sum of the subspaces associated to the disjuncts.

This raises a question concerning the connective "and": how is it to be interpreted? Also as the intersection of associated subspaces? To what other experiments could we apply the quantum modal logic presented here? It is well known that modalities are important in the discussions concerning the EPR paradox and the Bell theorem [2]: could the approach presented here be extended to this more general situation?

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